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Classification for the union of the totally geodesic 3-punctured spheres in an orientable hyperbolic 3-manifold

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1 Introduction

This note is a brief summary of my talk at RIMS workshop “Topology and Analysis of Discrete Groups and Hyperbolic Spaces”. I am preparing a paper [7] for details.

We consider essential 3-punctured spheres in orientable hyperbolic 3-manifolds. In this note, a hyperbolic 3-manifold is a manifold with toroidal boundary whose interior admits a complete hyperbolic metric. Its boundary component is called a cusp. A 3-punctured sphere is the compact orientable surface of genus zero with three boundary components. We always assume that the boundary of a 3-punctured sphere is contained in the cusps.

Adams [1] showed that an essential 3-punctured sphere in a hyperbolic 3-manifold is isotopic to a totally geodesic one. Hence we naturally assume that a 3-punctured sphere is totally geodesic.

We remark on immersed 3-punctured spheres.

Theorem 1.1 (Agol [2]). *Suppose that a totally geodesic 3-punctured sphere Σ is immersed in an orientable hyperbolic 3-manifold M , and Σ is not embedded. Then M is homeomorphic to a manifold obtained by (possibly empty) Dehn filling on a cusp of the Whitehead link complement. Furthermore, Σ is contained in M as in Figure 1.*

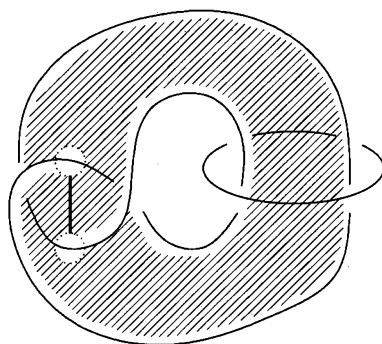


Figure 1: Non-embedded 3-punctured sphere

From now on, we consider embedded totally geodesic 3-punctured spheres in an orientable hyperbolic 3-manifold. 3-punctured spheres in a hyperbolic 3-manifold may intersect. The union of 3-punctured spheres may be complicated.

Theorem 1.2. *A connected component X of the union of all the totally geodesic 3-punctured spheres in an orientable hyperbolic 3-manifold with finitely many cusps is one of the following types:*

- (general types)
 $A_n(n \geq 1), B_{2n}(n \geq 1), T_3, T_4$.
- (types determining the manifolds)
 $Whi_{2n}(n \geq 2), Whi'_{4n}(n \geq 2), Bor_6, Mag_4, Tet_8, Pen_{10}, Oct_8$.
- (types almost determining the manifolds)
 $\widehat{Whi}_n(n \geq 2), \widehat{Whi}'_{2n}(n \geq 1), \widehat{Tet}_2, \widehat{Pen}_4, \widehat{Oct}_4$.

We classify the topological type of a pair $(N(X), X)$, where $N(X)$ is a regular neighborhood of X . The index indicates the number of 3-punctured spheres.

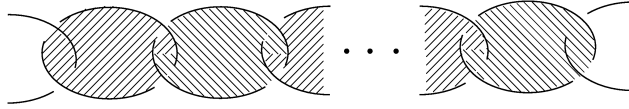


Figure 2: A_n

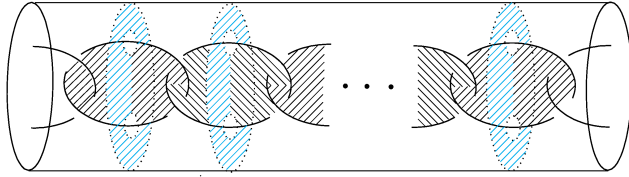


Figure 3: B_{2n}

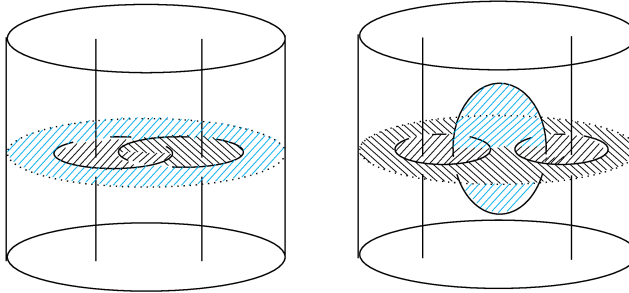
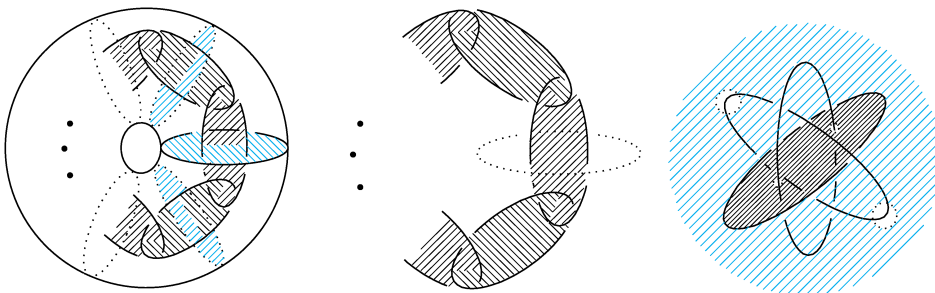
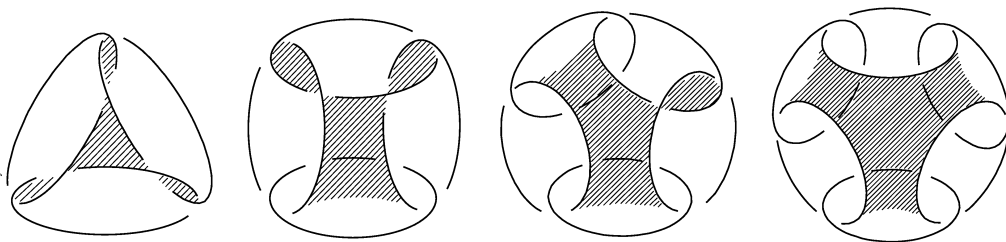
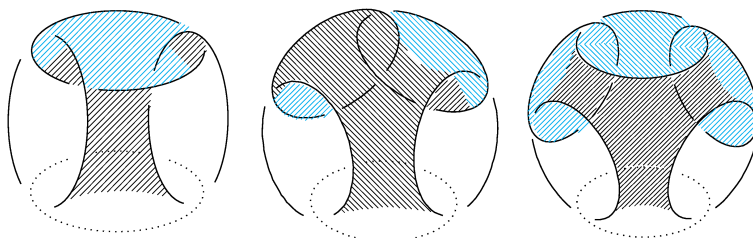


Figure 4: T_3 and T_4

Figure 5: Whi_{2n} (Whi'_{4n}), $\widehat{Whi_n}$ ($\widehat{Whi'_{2n}}$) and Bor_6 Figure 6: Mag_4 , Tet_8 , Pen_{10} and Oct_8 Figure 7: $\widehat{Tet_2}$, $\widehat{Pen_4}$ and $\widehat{Oct_4}$

The general types appear in various manifolds. For any finite multiset of general types, there is a manifold containing 3-punctured spheres of those types. When the type B_{2n} , T_3 or T_4 appears, however, there are isolated 3-punctured spheres near it. Contrastingly, each of the determining types appears only in a certain special manifold. The almost determining types appear only in manifolds obtained by Dehn fillings on a cusp of those special manifolds. For each n , the types Whi_{4n} and Whi'_{4n} have the common topology of the union, but they are distinguished by their neighborhood. The same argument holds for $\widehat{Whi_{2n}}$ and $\widehat{Whi'_{2n}}$.

We remark on non-hyperbolic 3-manifolds. If a 3-punctured sphere is contained in an irreducible 3-manifold whose boundary consisting of incompressible tori, it is cut into

a 3-punctured sphere and annuli by the JSJ decomposition. Hence we are reduced to the cases for geometric pieces. A 3-punctured sphere is contained only in three Seifert 3-manifolds, which are the fiber bundles of 3-punctured sphere over a circle. The product of a 3-punctured sphere and a circle contains infinitely many (pairwise non-isotopic) 3-punctured spheres.

2 Description of the types

At first, we introduce the manifolds containing the 3-punctured spheres of special types. Let \mathcal{W}_n denote the manifold in the left of Figure 5 which is an n -sheeted cyclic cover of the Whitehead link complement. Let \mathcal{W}'_n denote the manifold which is obtained a half twist along a certain 3-punctured sphere of \mathcal{W}_n . For odd n , the manifold \mathcal{W}'_n is homeomorphic to \mathcal{W}_n by reversing orientation. See Kaiser-Purcell-Rollins [4] for more details. The manifold \mathcal{W}'_2 is the Borromean rings complement. For $3 \leq n \leq 6$, let \mathcal{M}_n denote the minimally twisted hyperbolic n -chain link complement as in Figure 6. The manifold \mathcal{M}_3 is called the magic manifold. The manifold \mathcal{M}_4 has the smallest volume of the orientable hyperbolic 3-manifolds with 4 cusps [6]. The manifolds $\mathcal{M}_3, \mathcal{M}_5$ and \mathcal{M}_6 are conjectured to have smallest volume with respect to the number of the cusps. The 3-punctured spheres of the type $\widehat{Whi}_{2n}, \widehat{Whi}'_{4n}, \widehat{Bor}_6, \widehat{Mag}_4, \widehat{Tet}_8, \widehat{Pen}_{10}$ and \widehat{Oct}_8 are contained only in the manifolds $\mathcal{W}_n, \mathcal{W}'_{2n}, \mathcal{W}'_2, \mathcal{M}_3, \mathcal{M}_4, \mathcal{M}_5$ and \mathcal{M}_6 respectively.

The Borromean rings complement \mathcal{W}'_2 has six 3-punctured spheres. We can put the Borromean rings in such a manner that each component is in a plane in \mathbb{R}^3 . Then there are two 3-punctured spheres in the union of each plane and the infinite point as shown in the right of Figure 5.

The 3-punctured spheres of the type A_n are placed linearly, and can be considered as the most general type. If the 3-punctured spheres are placed cyclically, the union is the type \widehat{Whi}_n or \widehat{Whi}'_{2n} . Now consider the 3-punctured spheres of the type \widehat{Whi}_n or \widehat{Whi}'_{2n} . A regular neighborhood of the union of such 3-punctured spheres and the adjacent cusps is homeomorphic to \mathcal{W}_n or \mathcal{W}'_{2n} . Hence if a hyperbolic 3-manifold has such 3-punctured spheres, it is obtained by Dehn filling on a cusp of \mathcal{W}_n or \mathcal{W}'_{2n} . In fact, such a surgered hyperbolic 3-manifold except $\mathcal{M}_3, \dots, \mathcal{M}_6$ has no more 3-punctured spheres. The manifolds $\mathcal{M}_3, \dots, \mathcal{M}_6$ have more 3-punctured spheres as shown in Figure 6. Rotational symmetries give the remaining 3-punctured spheres.

The types $\widehat{Tet}_2, \widehat{Pen}_4$ and \widehat{Oct}_4 appear only in the manifolds obtained by Dehn fillings on a cusp of $\mathcal{M}_4, \mathcal{M}_5$ and \mathcal{M}_6 respectively, except $\mathcal{M}_3, \mathcal{M}_4$ and \mathcal{M}_5 . The 3-punctured spheres of $\widehat{Tet}_8, \widehat{Pen}_{10}$ and \widehat{Oct}_8 which are disjoint from the filled cusp remain in $\widehat{Tet}_2, \widehat{Pen}_4$ and \widehat{Oct}_4 .

We describe the remaining general types. Let \mathcal{T}_3 and \mathcal{T}_4 denote the hyperbolic 3-manifolds with totally geodesic boundary obtained by cutting \mathcal{M}_5 and \mathcal{M}_6 respectively along a 3-punctured sphere. Similarly, let \mathcal{B}_n denote the hyperbolic 3-manifold with totally geodesic boundary obtained by cutting \mathcal{W}_n along a 3-punctured sphere which intersects only one of the other 3-punctured spheres. The types $\mathcal{T}_3, \mathcal{T}_4$ and \mathcal{B}_{2n} appear in the manifolds containing $\mathcal{T}_3, \mathcal{T}_4$ and \mathcal{B}_{n+1} respectively. Hence if there are the 3-punctured

spheres of the type T_3, T_4 or B_{2n} , there are isolated 3-punctured spheres which correspond to the boundary of $\mathcal{T}_3, \mathcal{T}_4$ or \mathcal{B}_{n+1} .

We explain that the types Tet_8, Pen_{10} and Oct_8 are quite symmetrical in the way introduced by Dunfield-Thurston [3]. Each of $\mathcal{M}_4, \mathcal{M}_5$ and \mathcal{M}_6 has an involutorial symmetry as shown in Figure 8, 9 and 10. The quotients by this involution is naturally decomposed by ideal polyhedra. Then the original manifolds are recovered by double branched coverings. The quotient of \mathcal{M}_5 is the boundary of 4-dimensional simplex (a.k.a. pentachoron) made of five regular ideal tetrahedra. The quotient of \mathcal{M}_6 is the double of a regular ideal octahedron. The 3-punctured spheres in $\mathcal{M}_4, \mathcal{M}_5$ and \mathcal{M}_6 correspond to the faces of ideal polyhedra in this decomposition. In particular, the manifolds $\mathcal{M}_4, \mathcal{M}_5$ and \mathcal{M}_6 have symmetries which can map a 3-punctured sphere to any other one.

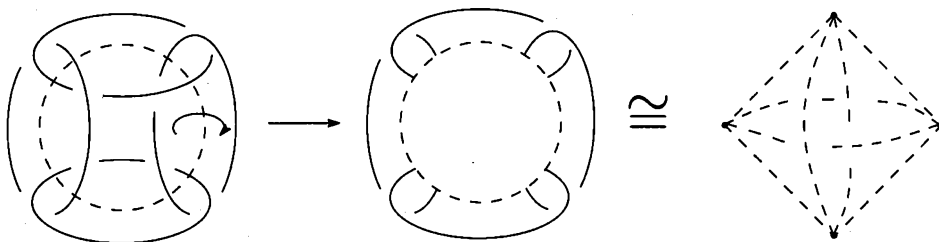


Figure 8: Quotient of \mathcal{M}_4

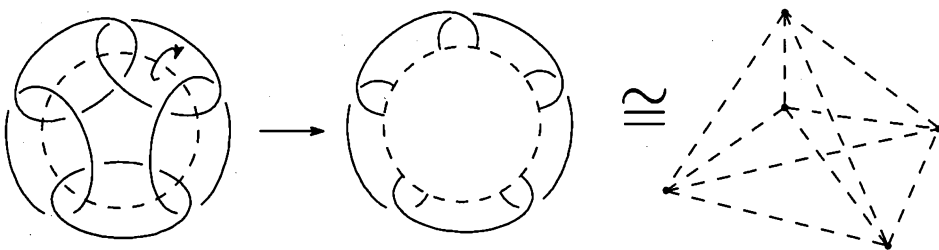


Figure 9: Quotient of \mathcal{M}_5

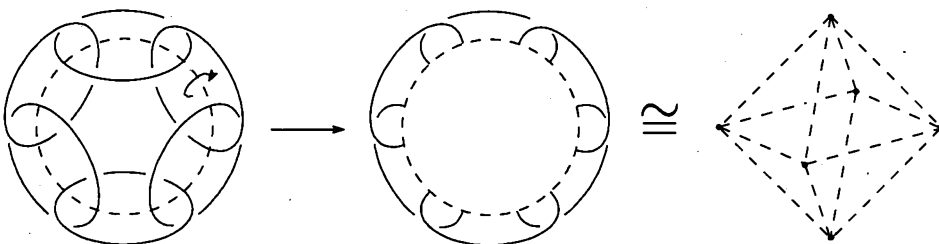


Figure 10: Quotient of \mathcal{M}_6

3 Number of 3-punctured spheres

For a hyperbolic 3-manifold with totally geodesic boundary, we can estimate its volume by the Euler characteristic of its boundary.

Theorem 3.1 (Miyamoto [5]). *Let N be a hyperbolic manifold with totally geodesic boundary. Then $\text{vol}(N) \geq |\chi(\partial N)|V_{\text{oct}}/2$, where V_{oct} is the volume of a regular ideal octahedron.*

When we cut a hyperbolic 3-manifold along totally geodesic surfaces, we obtain a hyperbolic manifold with totally geodesic boundary. Hence if a hyperbolic 3-manifold M contains n disjoint 3-punctured spheres, the volume of M is at least nV_{oct} . Therefore the classification gives an estimate of the number of 3-punctured spheres by the volume. For example, 3-punctured spheres of the type A_n contains $\lfloor (n+1)/2 \rfloor$ disjoint ones.

Corollary 3.2. *Suppose that an orientable hyperbolic 3-manifold M has n 3-punctured spheres. Then $n \leq 4\text{vol}(M)/V_{\text{oct}}$. The equality holds if and only if M is the manifold M_4 .*

4 Sketch of proof

There are six simple geodesics in a totally geodesic 3-punctured sphere. Each of these six geodesics is one of the two types. We indicate a geodesic whose endpoints are in different boundary components by the type α . Let the others be of the type β . At first, we consider the intersection of two 3-punctured spheres.

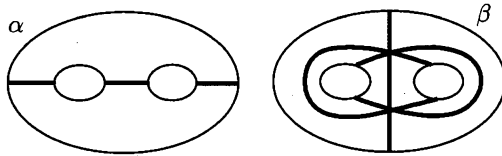


Figure 11: Simple geodesics in a 3-punctured sphere

Lemma 4.1. *Intersection of two totally geodesic 3-punctured spheres in an orientable hyperbolic 3-manifold is one of the following types:*

- (o) *The intersection is empty, i.e. the two 3-punctured spheres are disjoint.*
- (i) *The intersection is one simple geodesic, and it is of the type α in both 3-punctured spheres.*
- (ii) *The intersection is one simple geodesic, and it is of the type α in a 3-punctured sphere and of type β in the other 3-punctured sphere.*
- (iii) *The intersection is two simple geodesics, and they are of the type α in both 3-punctured spheres.*

The orientability is necessary for Lemma 4.1. For instance, a non-orientable hyperbolic 3-manifold obtained by gluing one regular ideal octahedron contains two 3-punctured spheres which intersect at three geodesics.

In the case (iii), the union of the two 3-punctured spheres is $\widehat{Whi}_2, \widehat{Whi}'_2$ or \widehat{Tet}_2 as in Figure 12. The boundary of a regular neighborhood of these two 3-punctured spheres and the adjacent cusps is a torus. Hence the ambient 3-manifold is obtained by (possibly empty) Dehn filling on a cusp of $\mathcal{W}_2, \mathcal{W}'_2$ or \mathcal{M}_4 . Then the union of the 3-punctured spheres is $\widehat{Whi}_2, \widehat{Whi}_4, \widehat{Whi}'_2, \widehat{Bor}_6, \widehat{Tet}_2, \widehat{Mag}_4$ or \widehat{Tet}_8 .

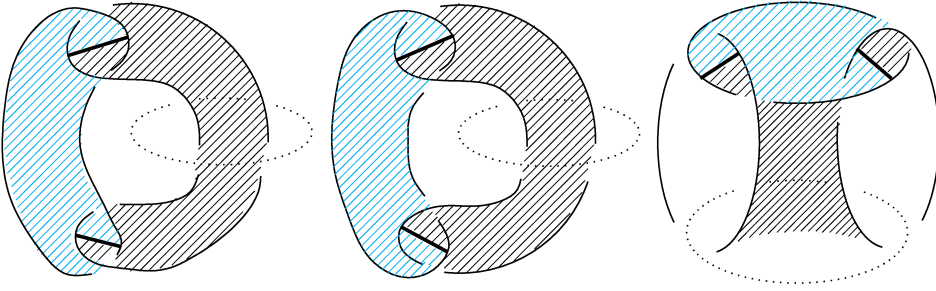


Figure 12: Two 3-punctured spheres intersecting at two geodesics

We next consider the case (ii). Suppose that 3-punctured spheres Σ_1 and Σ_2 intersect at a geodesic of the type β in Σ_1 and of the type α in Σ_2 . Their union is B_2 . If another 3-punctured sphere intersects Σ_2 , the ambient 3-manifold is the Borromean rings complement \mathcal{W}'_2 . Otherwise the component of the union containing Σ_1 and Σ_2 is B_{2n}, Whi_{2n} or Whi'_{4n} .

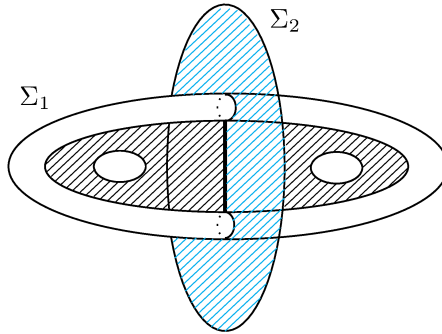


Figure 13: Two 3-punctured spheres of the type B_2

From now on, we consider a component X of the union without intersection of the type (ii) or (iii). Consider intersection C of a cusp and X . We can assume that a slope of C is 0, 1 or ∞ . Remark that there are only two geodesic of type α emanating from one component of boundary. We first consider following special types of intersection:

- If C contains three loops of slope 0, 1 and ∞ with common intersection, X contains T_3 .
- If C contains three loops of slope 0, 1 and ∞ without common intersection, X contains \widehat{Pen}_4 .
- If C contains four loops of slope 0 and ∞ , X contains \widehat{Oct}_4 .

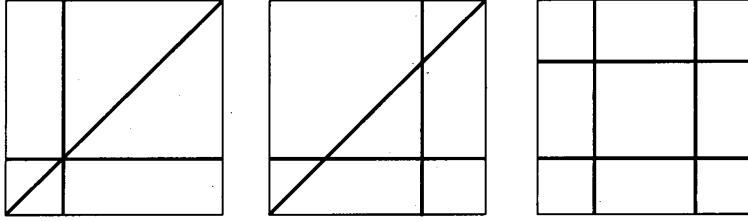


Figure 14: Special types of C (The squares are fundamental domains of cusps.)

These can be constructed explicitly. In these cases, the union X is T_3 , \widehat{Pen}_4 , \widehat{Oct}_4 , Oct_8 or Pen_{10} . All the three above cases hold for Pen_{10} .

We next consider the remaining cases. Then the intersection C is one of the following general types;

- disjoint simple loops,
- two loops with one common point, or
- three loops with two common points, two of which are parallel.

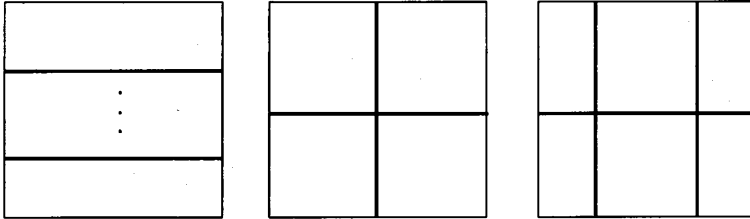


Figure 15: General types of C

If there is a 3-punctured sphere which intersects three 3-punctured spheres, the union X is T_4 . Then they intersect no more 3-punctured spheres.

In the last remaining cases, the 3-punctured spheres in X form a line or a cycle. Then the union X is A_n , \widehat{Whi}_n or \widehat{Whi}'_{2n} .

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